Some probabilistic aspects of fractal growth

1987 J. Phys. A: Math. Gen. 20 5025

(http://iopscience.iop.org/0305-4470/20/14/039)

J. Phys. A: Math. Gen. 20 (1987) 5025-5028. Printed in the UK

COMMENT

Some probabilistic aspects of fractal growth

A A Tsonis

Department of Geological and Geophysical Sciences, The University of Wisconsin-Milwaukee, Milwaukee, WI 53201, USA

Received 4 March 1987

Abstract. Non-equilibrium growth models generate fractal structures that mimic phenomena observed in nature. In this comment simulations indicate that these generated structures may also be the most probable events.

Non-equilibrium growth results in structures which appear random and amorphous. Dielectric breakdown and dendritic growth are two very familiar examples of non-equilibrium growth. Lately, the introduction of simple non-equilibrium models and the utilisation of fractal geometry [1] has led to a good understanding of the morphology of such growth. We now know that there is 'order' in those disorderly appearing structures. This order is called scale invariance. A scale-invariant structure is an object whose statistical properties are unchanged under a change of spatial length scale. Scale-invariant structures are also called fractals because their Hausdorff-Besicovitch dimension is not an integer. In other words, the total mass of a fractal object, M, scales with some characteristic length, l, as $M(l) \propto l^D$ where D is not an integer. The existence of D indicates that there is a relation between properties at different scales and thus 'order'. Non-equilibrium growth models were initially proposed by Witten and Sander [2], according to which structures with a well defined fractal geometry are generated by a simple diffusion-limited aggregation (DLA) process in which particles are added one at a time to a growing cluster via random walk trajectories. This model generated a great interest and led to the development of a variety of related models that simulate phenomena such as dielectric breakdown [3], lightning [4], viscous fingering [5], the development of morphology in biological systems [6], dendritic growth [7], etc. The type of model which is of interest in this comment is the dielectric breakdown model (DBM) of Niemeyer *et al* [3]. According to this model, at each step the electric potential is determined for all perimeter sites of the evolving discharge pattern. Then every perimeter site is associated with a 'growth' probability which is proportional to the local electric field. The assigned probabilities thus determine at each step a probability distribution. Given this probability distribution a site is selected randomly and added to the evolving pattern. The above procedure is then repeated until a large structure is obtained. The above simulations which are performed on a square lattice placing the growth site at the centre can also be modified to simulate lightning in the atmosphere [4]. These simulations are carried out on a strip geometry where the initial growth site is placed at the centre of one side of a rectangular box. An example of such a simulated lightning is shown in figure 1. The structure is fractal with a fractal dimension equal to 1.36 [4].

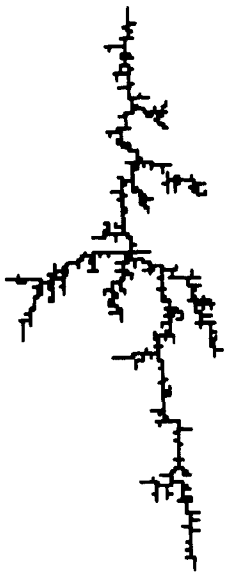

Figure 1. Example of computer-generated lightning. This dielectric breakdown simulation was carried out on a 250×150 rectangular lattice, where the initial growth site was placed at the centre of the top side. For more details see [4]. The structure consists of 736 points and its fractal dimension is 1.36. The vertical extent of this structure is 150 lattice units.

Since the model provides the probability of every site at each step one may calculate the probability of the whole structure by multiplying the probabilities which all the selected candidates were associated with. In general, if we assume that at the nth step a structure is made up from n points denoted as $A_1, A_2, \ldots, A_n$ which are selected in that order then the probability, $P(n)$, of this structure is given by

$$P(n) = P(A_1, A_2, \ldots, A_n)$$
$$= P(A_1)P(A_2|A_1)P(A_3|A_1A_2) \ldots P(A_n|A_1A_2 \ldots A_{n-1}).$$

According to dielectric breakdown type models the tip of a line has the highest probability. Therefore, for model simulations on a strip geometry, a straight line is the structure with the maximum growth probability. In other words a straight line is the most probable outcome of such a model. Such an outcome, however, is never achieved, just as we never see a straight line of lightning in the atmosphere. In a perfectly uniform atmosphere one would expect the breakdown to spread out in a straight line. However, because of noise in the system a growth instability will occur and irregularities will appear [7, 8]. In the atmosphere such noise can be density, temperature or humidity fluctuations. This noise is reproduced in the dielectric breakdown type models via the random selection procedure at each step. No matter how small the growth probability of a site is, there is always a chance that that site will be selected. Thus, the evolving structure soon becomes very irregular. The interesting point, however, is that the irregular evolution of such structure is characterised by properties which are related at different scales and that all structures generated by such models have a reproducible fractal dimension. In view of the foregoing discussion the following question arises: are fractals more probable events? If fractal structures are most probable events, then one would expect the chance of getting a non-fractal discharge pattern out of a DBM to be very, very small. A theoretical way to prove this does not exist and to demonstrate this by simulating structures, until a non-fractal one is generated, is hopeless due to the probabilities involved. The best alternative is to 'cook up' a non-fractal discharge pattern which consists of n points, 'force' the model to generate it and then compare it with a fractal structure which was normally generated by the model and which consists of n points as well. This means that if at each step the model does not select one of the candidates that fits the non-fractal structure the selection procedure is repeated until a point that fits that structure is selected. Such a non-fractal discharge pattern is shown in figure 2. It consists of, approximately, an equal number of 'points' as the fractal structure in figure 1 and its probability as a function of the step is shown in figure 3 by the broken curve. The full curve indicates

Figure 2. A hypothetical non-fractal discharge pattern. This structure consists of 738 points. The vertical extent of this structure is 150 lattice units. The mass (number of occupied lattice sites) (M) of this structure scales with distance (r) from the origin as $M(r) \propto r^D$ where $D = 1$. Therefore this structure is not a fractal [1].

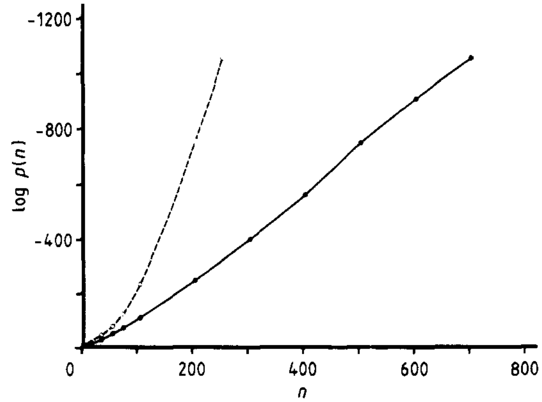

Figure 3. The probability of an evolving structure as a function of the step. The full curve refers to the structure in figure 1 and the broken curve refers to the structure in figure 2.

the corresponding probability for the structure in figure 1. As can be observed in figure 3, the probability of the non-fractal structure is much, much smaller than that of the fractal structure and it becomes even smaller as the number of steps increases. After only 70 steps the probability of the fractal structure is greater by a factor of about 10^{40}. After 200 step this factor has become 10^{500}.

In view of the above results the following interesting questions may be posed. Are those differences observed because the generation of the non-fractal structure was 'forced'? Will those differences be observed if the model is forced to generate a given fractal structure (such as the one shown in figure 1)? These questions can be answered if we answer a more general question. Obviously, a given structure may be generated in many different ways (the one we force will be one of the possible ways). Does the total probability, $P(n)$, of a given structure depend on the order in which it is built up? Experimentation with the structures in figures 1 and 2 indicates that when a structure is 'young' ($n \leqslant 30$) significant differences in $P(n)$ may be observed. These differences diminish rapidly and for a given structure and a given $n \geqslant 70$, $P(n)$ has a reproducible value which does not depend on the order in which the structure is built up. Therefore, the probability of the fractal structure will always be infinitely higher than the probability of the non-fractal structure, no matter how these structures are generated.

The above findings are confirmed by additional experimentation with fractal and non-fractal structures. Apparently, with n points one may produce a great number of different structures. Some of them will be more probable than others. From the more probable ones the most probable will most likely be generated. The above results demonstrate that the 'chosen' structure will be a fractal structure.

Basically, the results reported above are related to thermodynamics. At a finite temperature, T, the free energy is minimised. The free energy, F, is defined as $F = E - TS$ where E is the energy and S is the entropy. Thermodynamic probabilities are proportional to $\exp(-E/k_B T)$ and the entropy is proportional to the number of different configurations for the same energy. For larger structures the entropy is larger and thus the most probable solution which corresponds to lowest energy (in our case, a straight line of lightning) is not actually found. Apparently, very low probability solutions (ordered states) will not be found either. The larger variety of configurations in energetically less favourable states makes disordered states (fractals) appear more often. Only for small structures does one sometimes see the lowest energy state reached at a finite temperature. An interesting question which remains open is whether the numerical probabilities which are calculated from the model can be quantified by suitable power laws. Work in this area is in progress.

I thank James B Elsner for computing assistance.

References

[1] Mandelbrot B B 1977 *Fractals: Form, Chance and Dimension* (San Francisco: Freeman)
[2] Witten T A and Sander L A 1981 *Phys. Rev. Lett.* **47** 1499-501
[3] Niemeyer L, Pietronero L and Wiesmann H J 1984 *Phys. Rev. Lett.* **52** 1033-6
[4] Tsonis A A and Elsner J B 1987 *Beitr. Phys. Atmos.* to be published
[5] Nittmann J, Daccord G and Stanley H E 1985 *Nature* **314** 141-4
[6] Meakin P 1986 *J. Theor. Biol.* **118** 101-13
[7] Nittmann J and Stanley H E 1986 *Nature* **312** 663-8
[8] Saffman P G and Taylor G I 1958 *Proc. R. Soc.* A **245** 312-29